

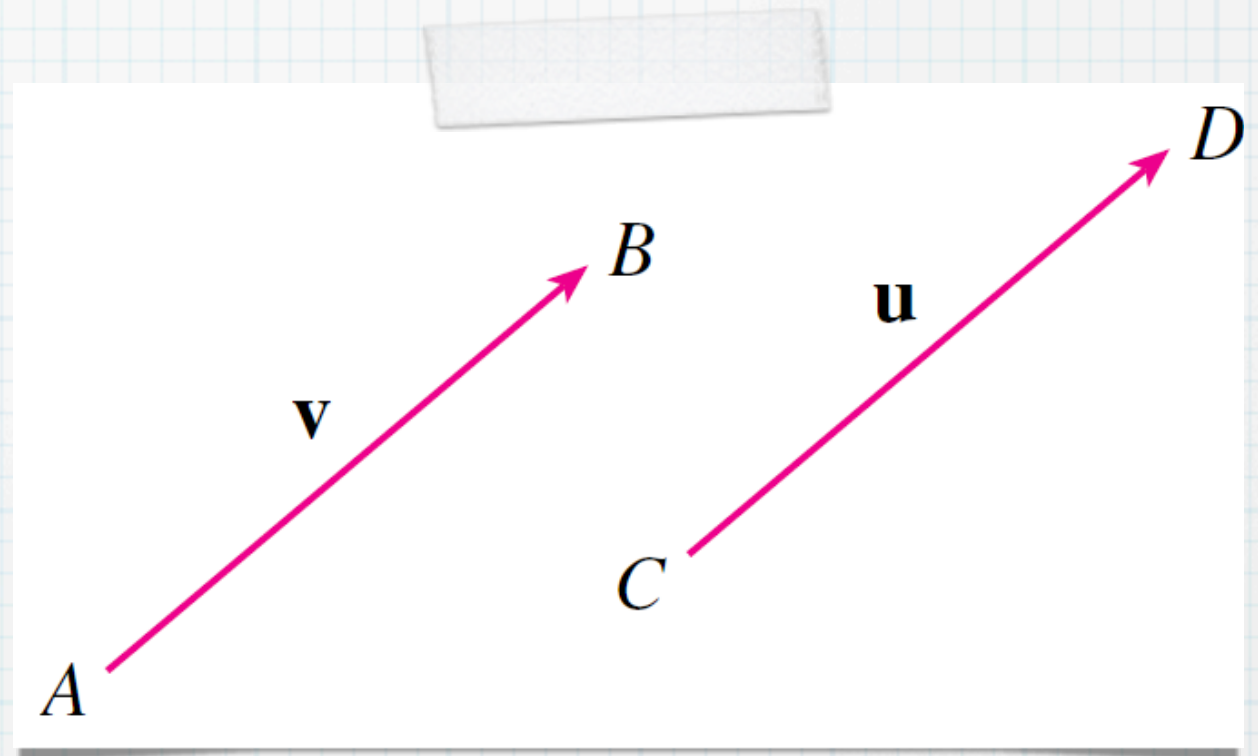
# Vectors

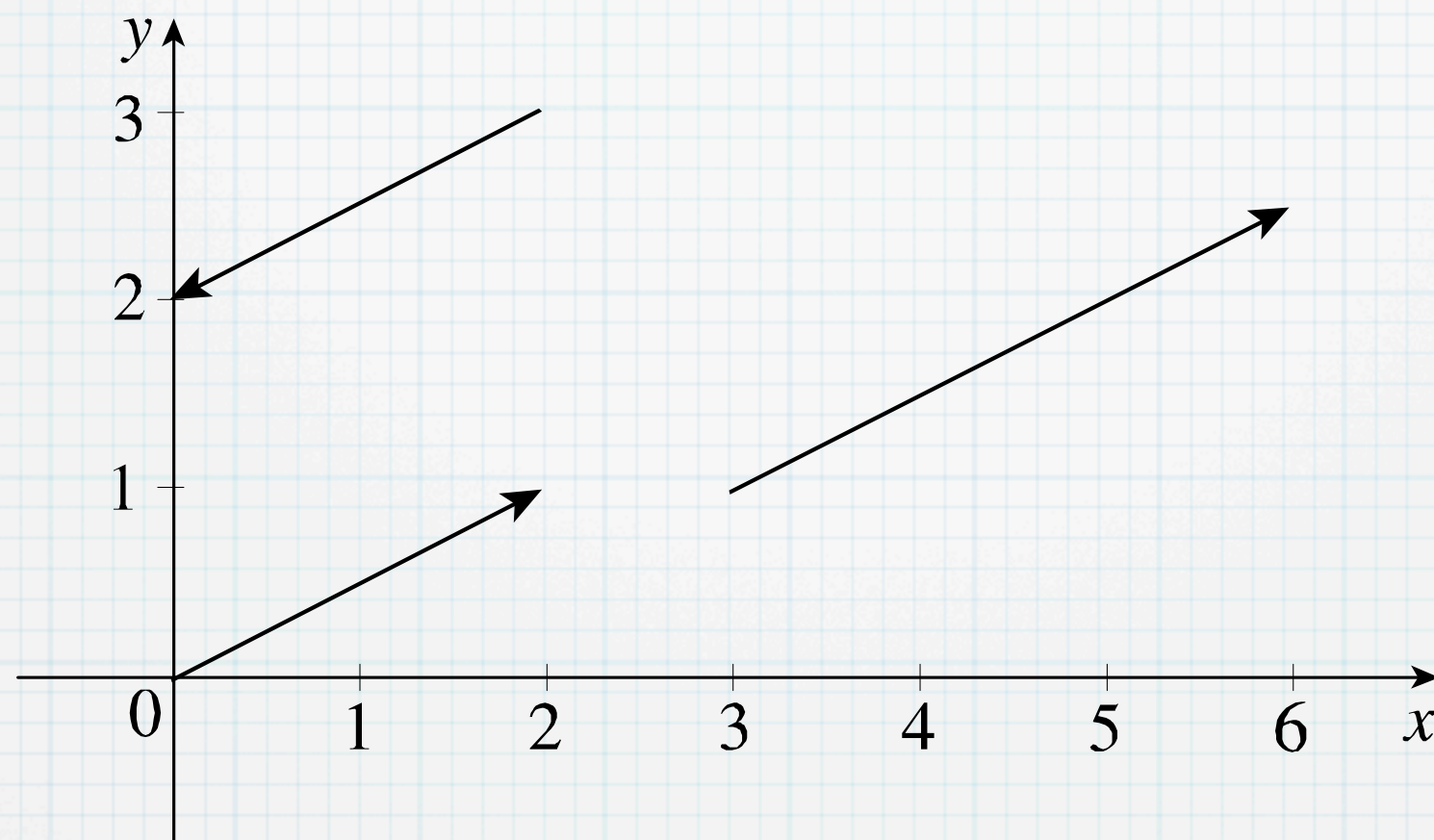
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## Section 9.2

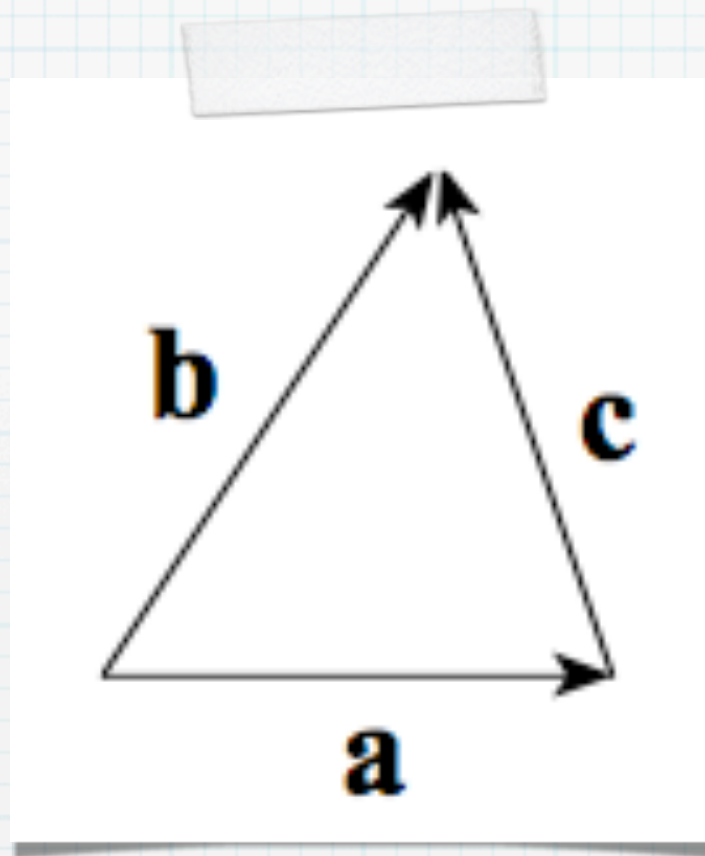
# Vectors

- \* A vector is a quantity (such as displacement or velocity or force) that has both magnitude and direction.
- \* A vector is often represented by an arrow or a directed line segment.
- \* The length of the arrow represents the magnitude of the vector and
- \* the arrow points in the direction of the vector.



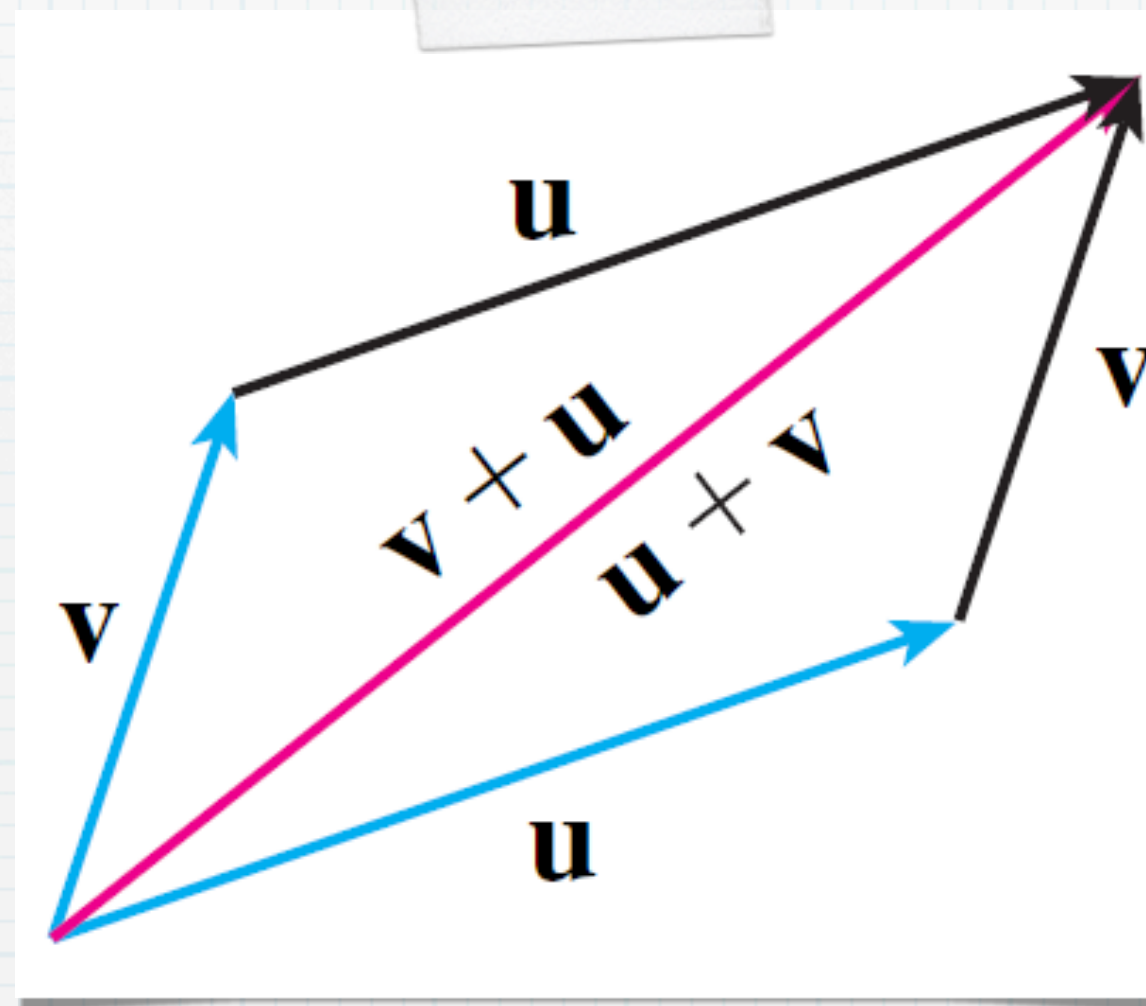


Parallel?



Describe  $c$  in terms of  $a$  and  $b$



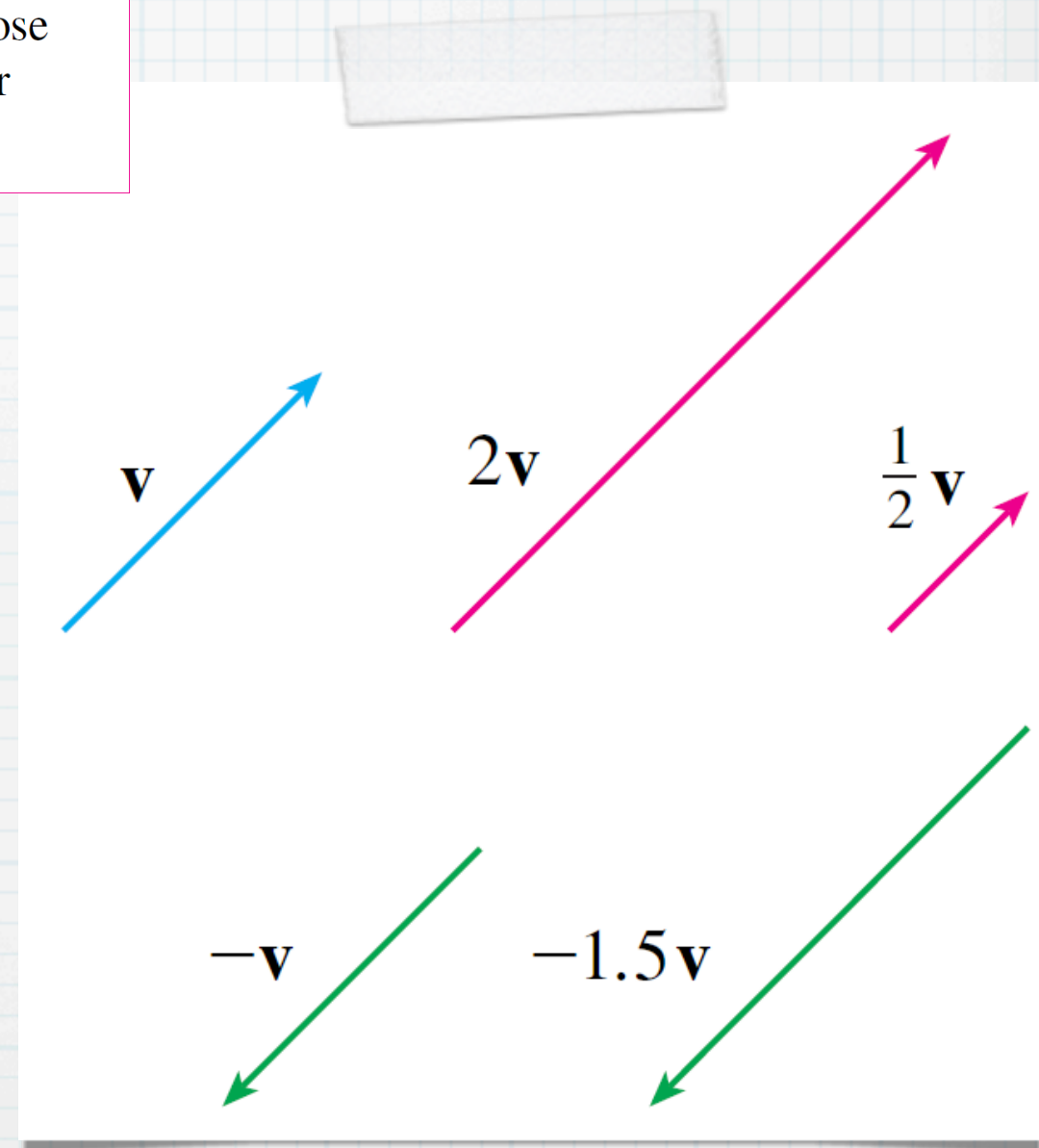


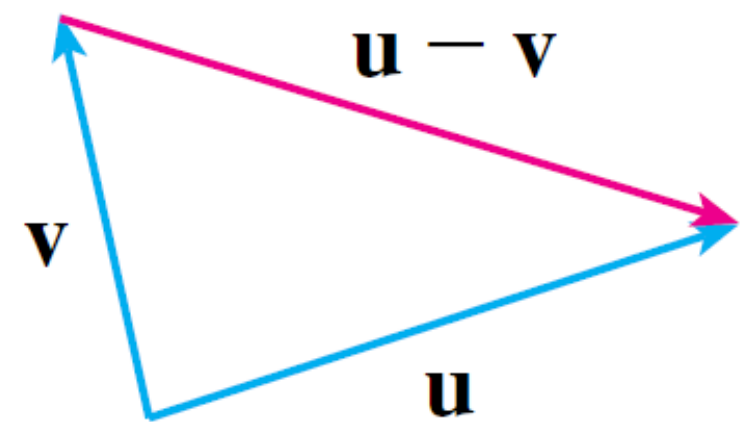
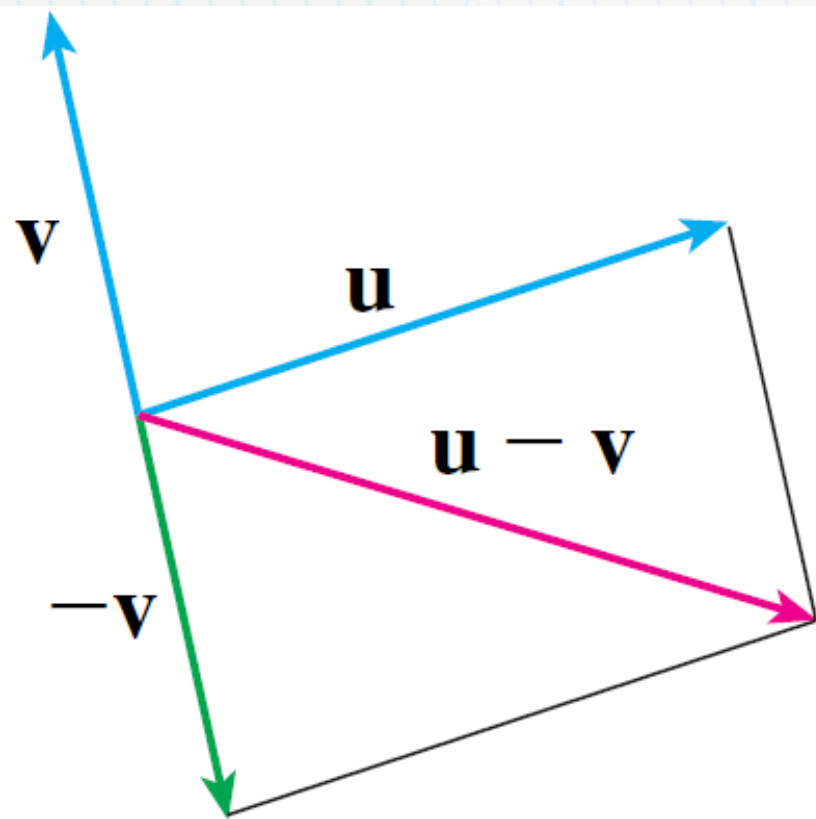
# Vector Addition

**Definition of Vector Addition** If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors positioned so the initial point of  $\mathbf{v}$  is at the terminal point of  $\mathbf{u}$ , then the **sum**  $\mathbf{u} + \mathbf{v}$  is the vector from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .

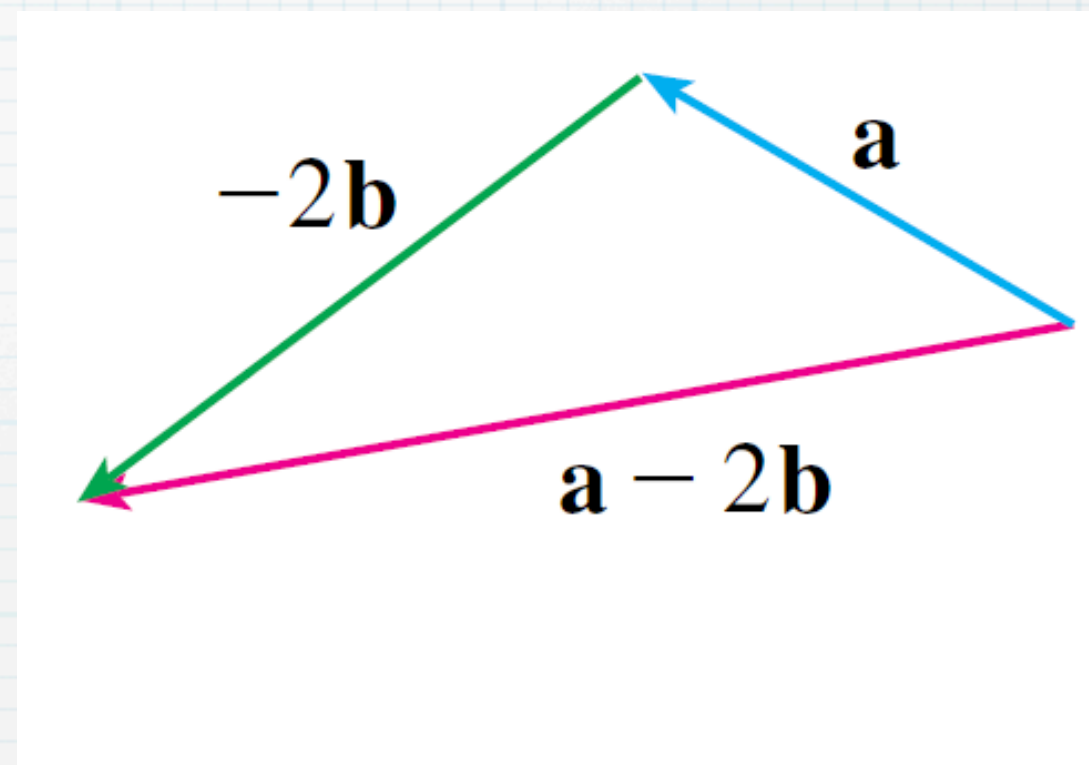
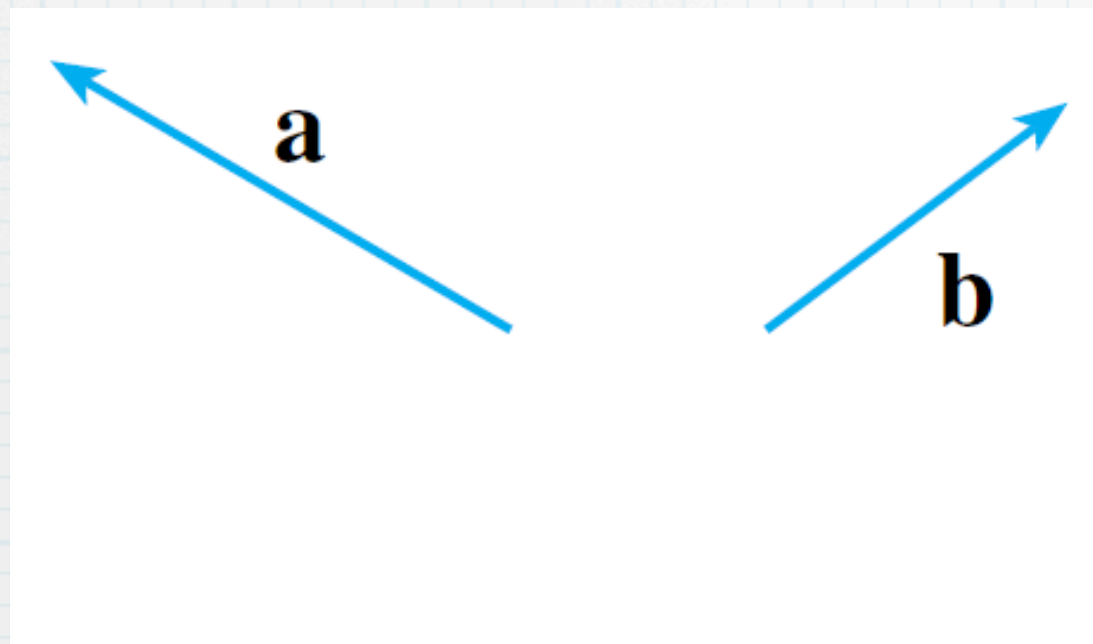
# Scalar Multiplication

**Definition of Scalar Multiplication** If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector whose length is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$ . If  $c = 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .





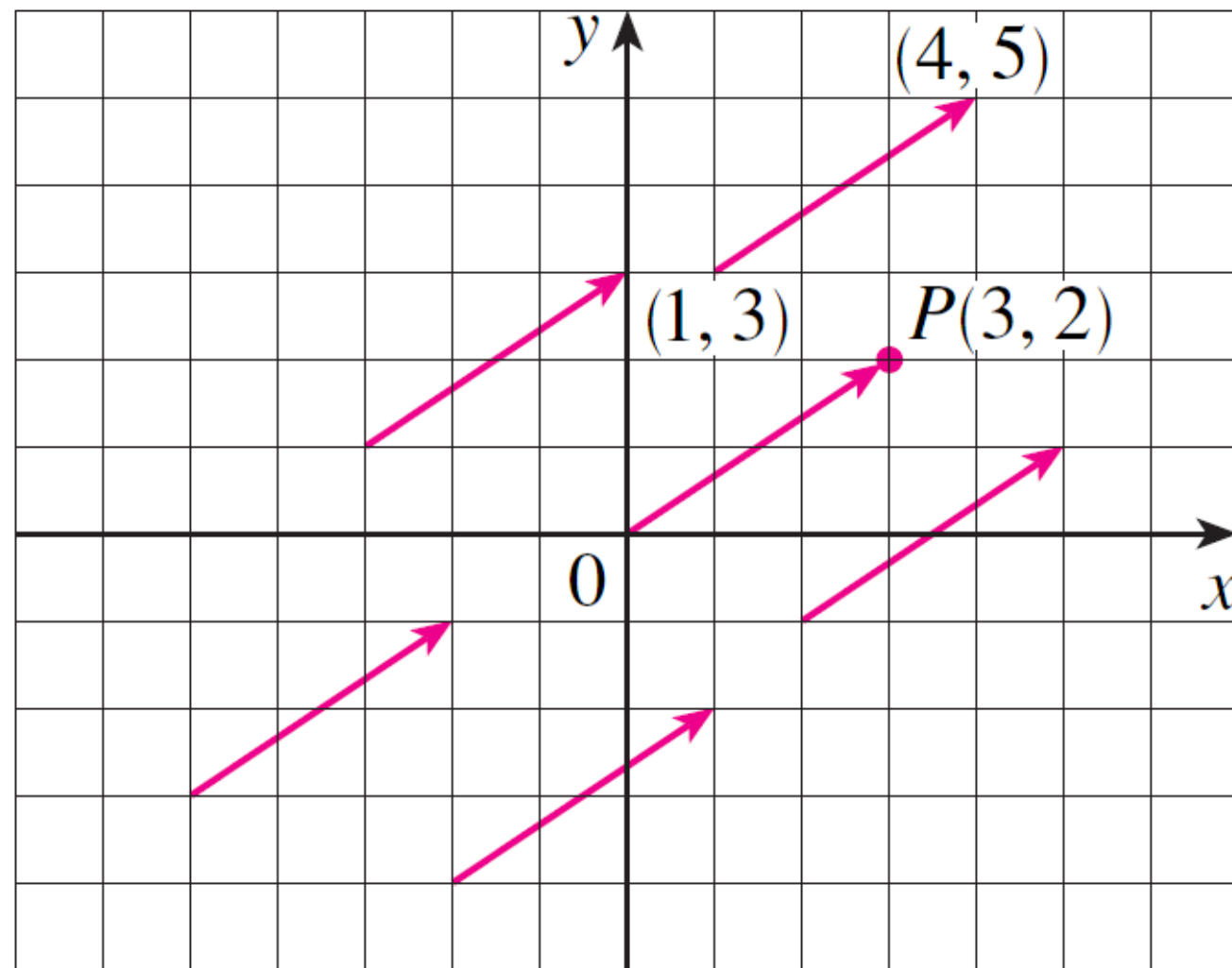
# Vector Subtraction





# Components

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$



Representations of the vector  $\mathbf{a} = \langle 3, 2 \rangle$

# Length

The length of the two-dimensional vector  $\mathbf{a} = \langle a_1, a_2 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

# Using Components

If  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

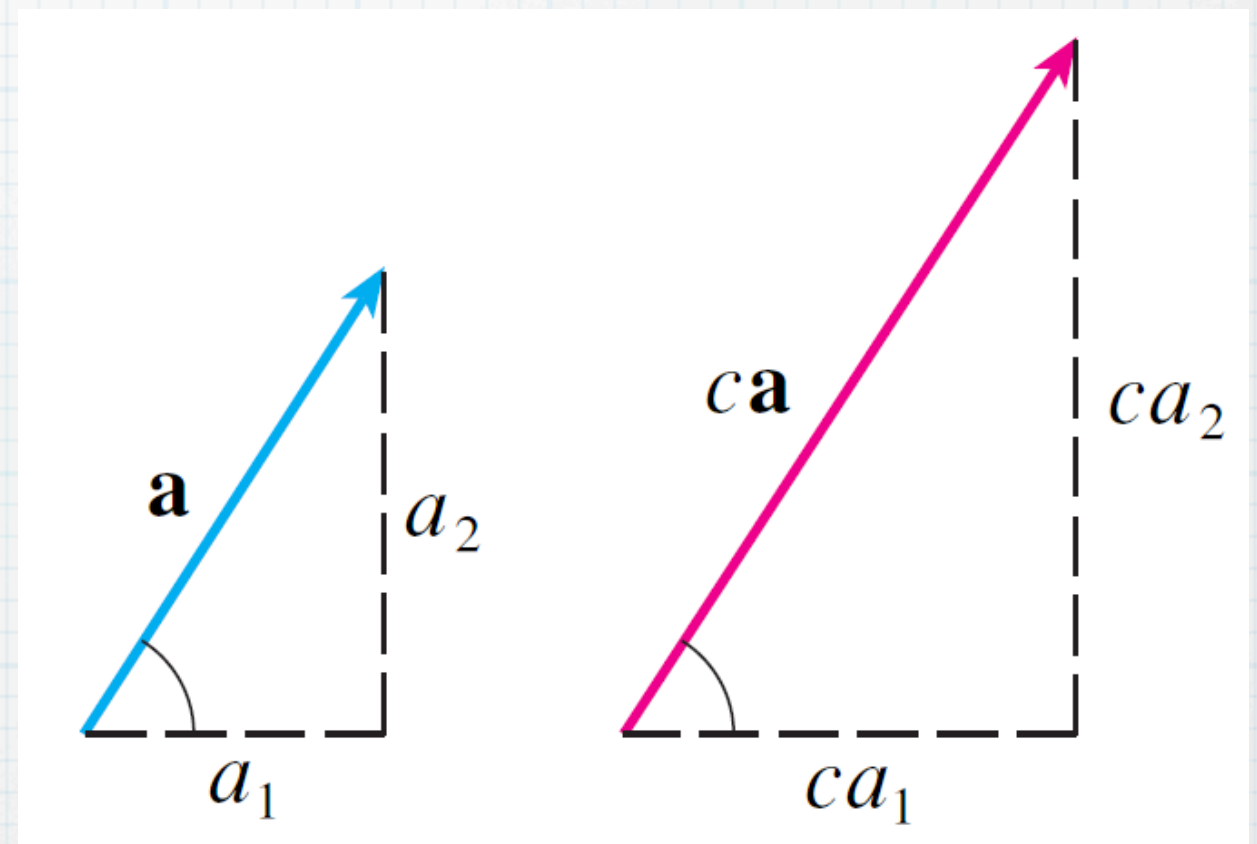
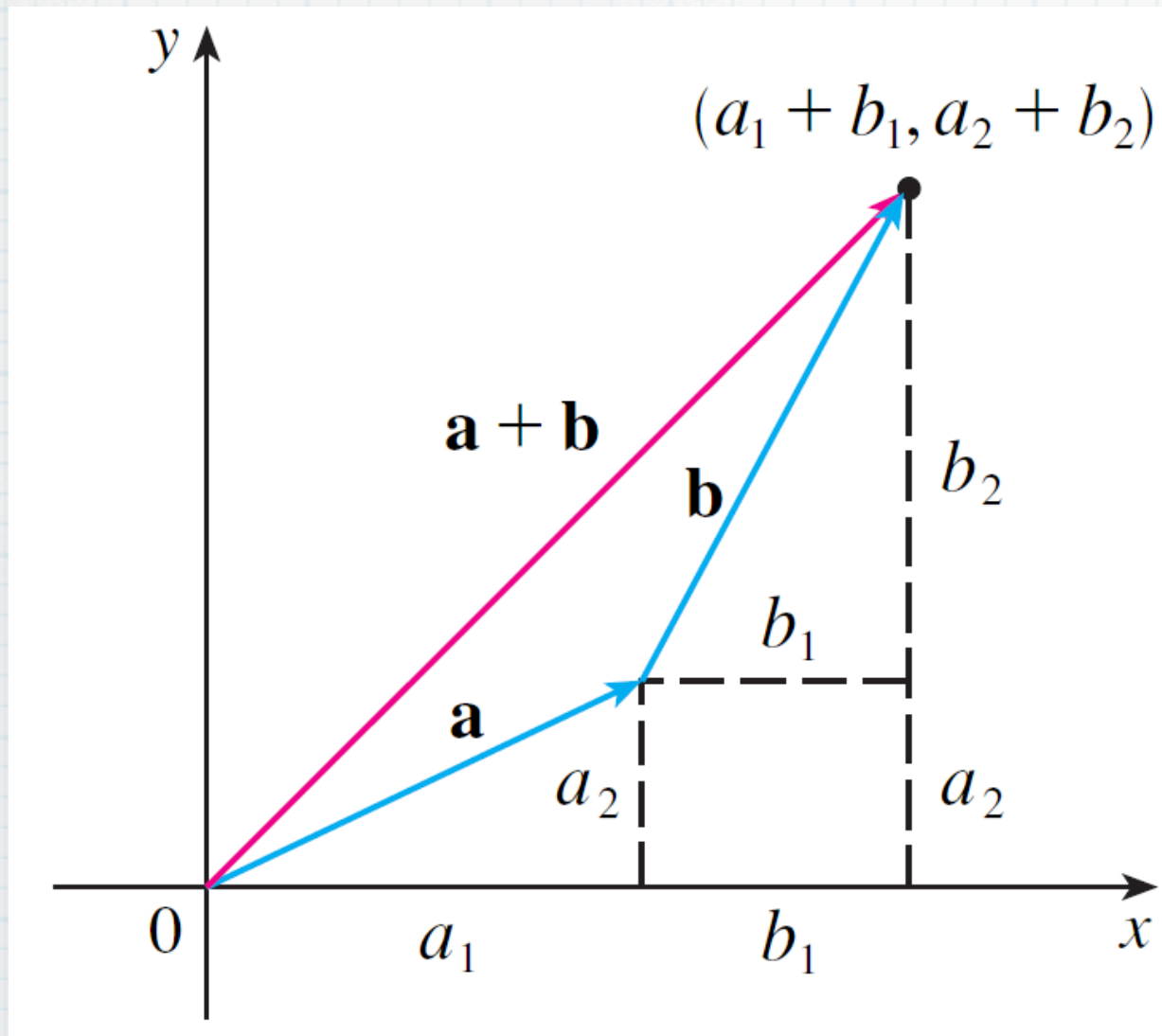
Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

# Graphically





# Properties of Vectors

**Properties of Vectors** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars, then

1.  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

2.  $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$

3.  $\mathbf{a} + \mathbf{0} = \mathbf{a}$

4.  $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$

5.  $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

6.  $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$

7.  $(cd)\mathbf{a} = c(d\mathbf{a})$

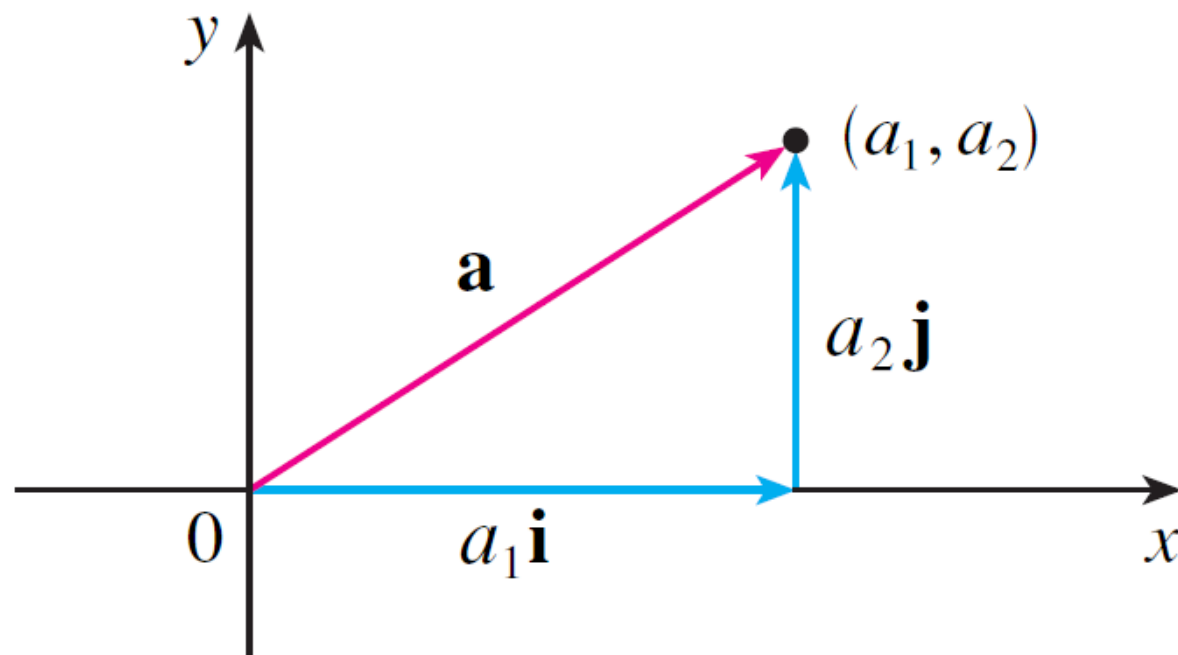
8.  $1\mathbf{a} = \mathbf{a}$

# Standard Bases

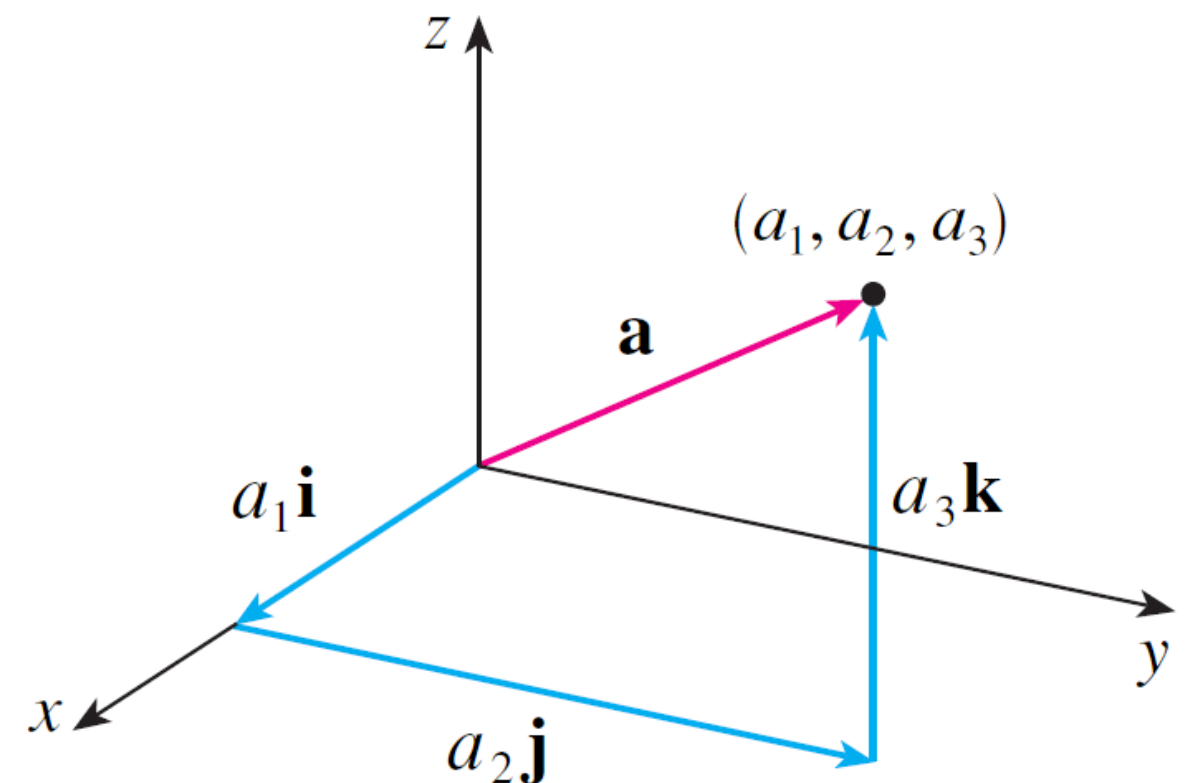
$$\mathbf{i} = \langle 1, 0, 0 \rangle$$

$$\mathbf{j} = \langle 0, 1, 0 \rangle$$

$$\mathbf{k} = \langle 0, 0, 1 \rangle$$



$$(a) \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$$



$$(b) \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

# Handout

- \* Component Vectors
- \* Where do they point? Group Work 3

# Dot Product

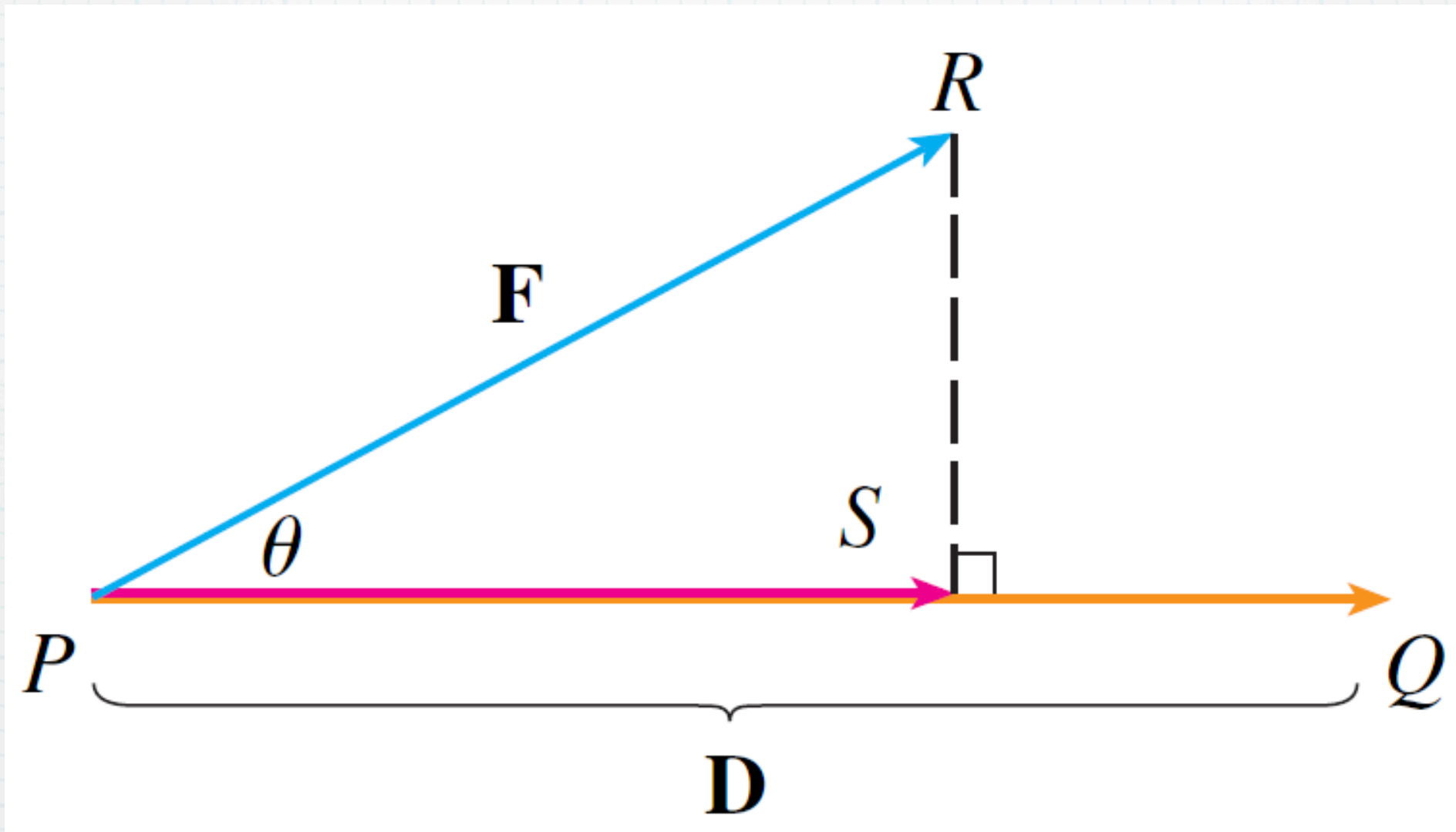
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## Section 9.3



# Work

Work done by a constant force  $F$  (in the line of motion) in moving an object through a distance  $d$ :  $W = Fd$



Need to know the component of the force  $F$  in direction of motion

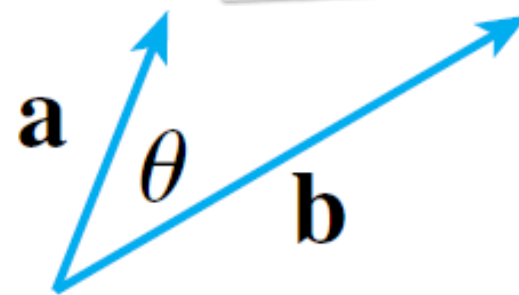
# Dot Product

**Definition** The **dot product** of two nonzero vectors **a** and **b** is the number

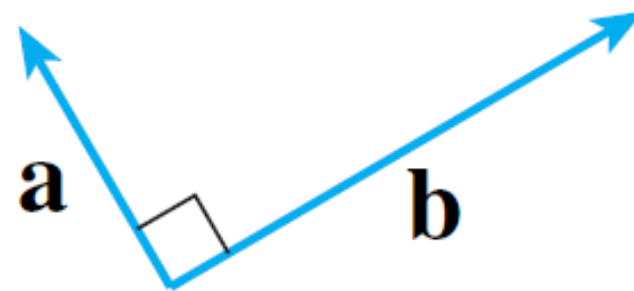
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

where  $\theta$  is the angle between **a** and **b**,  $0 \leq \theta \leq \pi$ . (So  $\theta$  is the smaller angle between the vectors when they are drawn with the same initial point.) If either **a** or **b** is **0**, we define  $\mathbf{a} \cdot \mathbf{b} = 0$ .

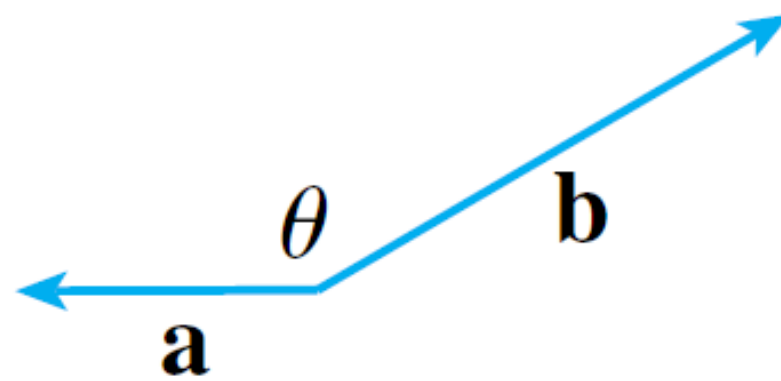
**2** Two vectors **a** and **b** are orthogonal if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$ .



$$\mathbf{a} \cdot \mathbf{b} > 0$$



$$\mathbf{a} \cdot \mathbf{b} = 0$$



$$\mathbf{a} \cdot \mathbf{b} < 0$$

# Dot Product: Component Form

The dot product of  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1) = 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\frac{1}{2} \rangle = (-1)(6) + 7(2) + 4(-\frac{1}{2}) = 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1) = 7$$



# Properties

**Properties of the Dot Product** If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_3$  and  $c$  is a scalar, then

1.  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

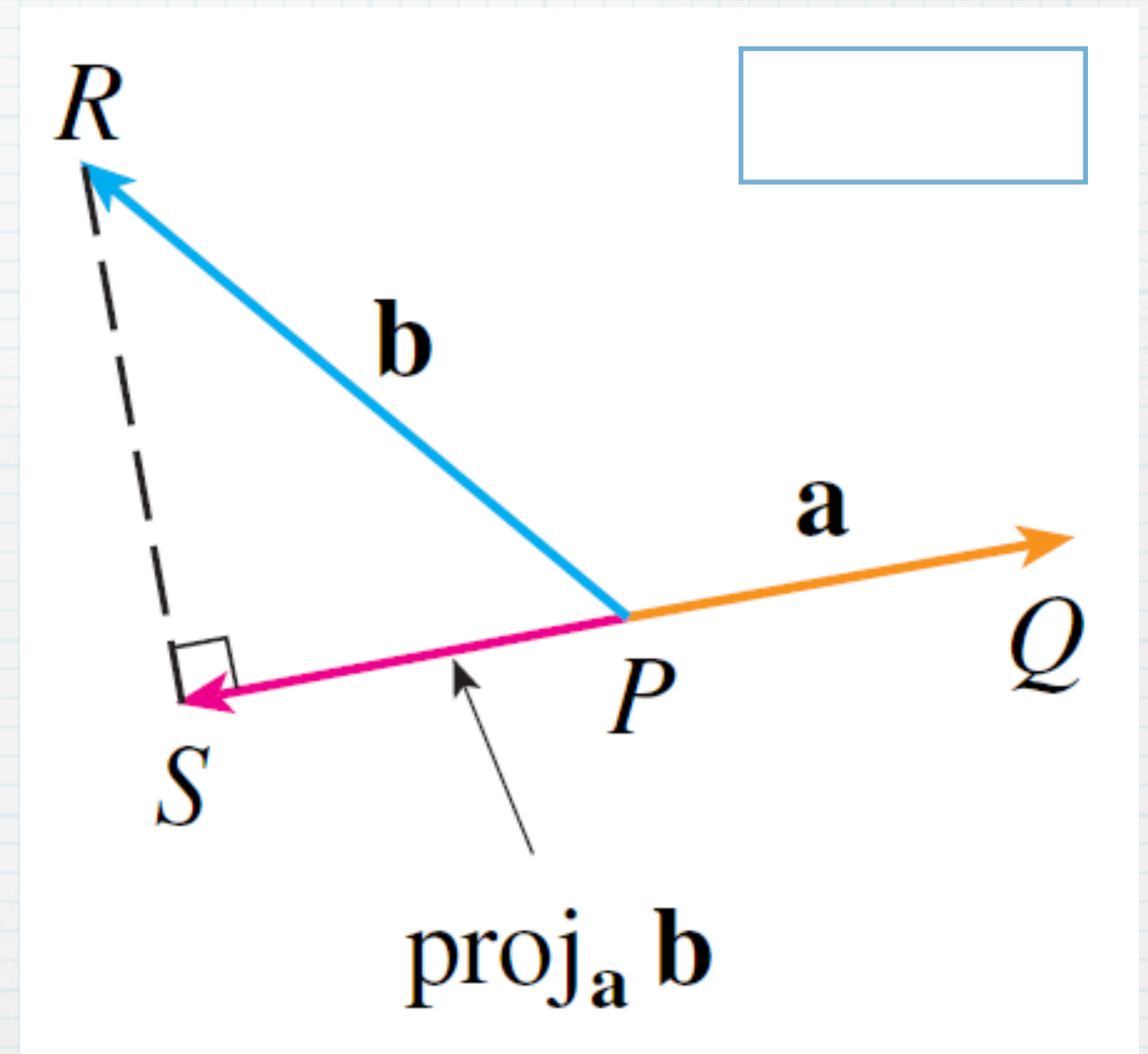
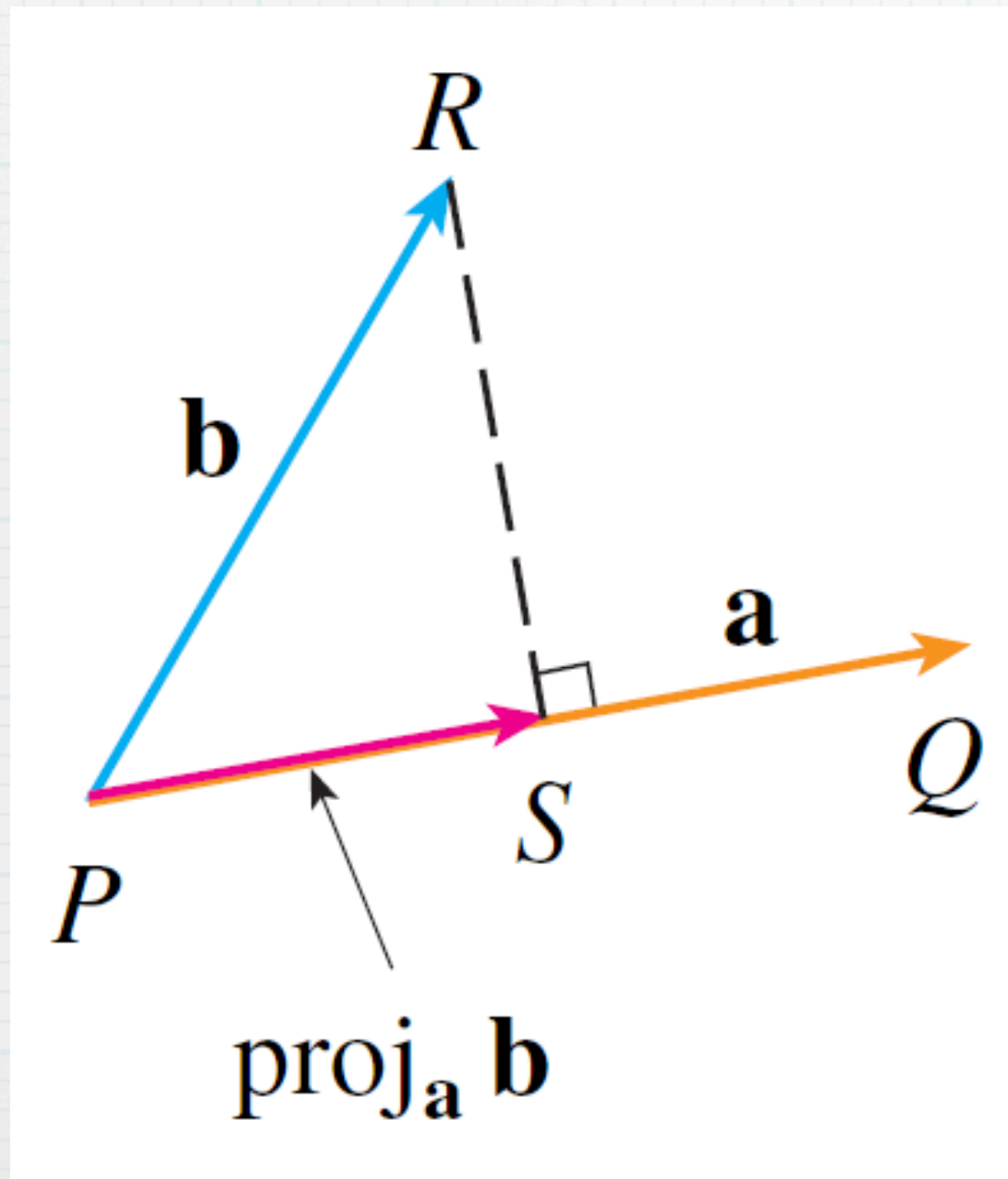
2.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

3.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

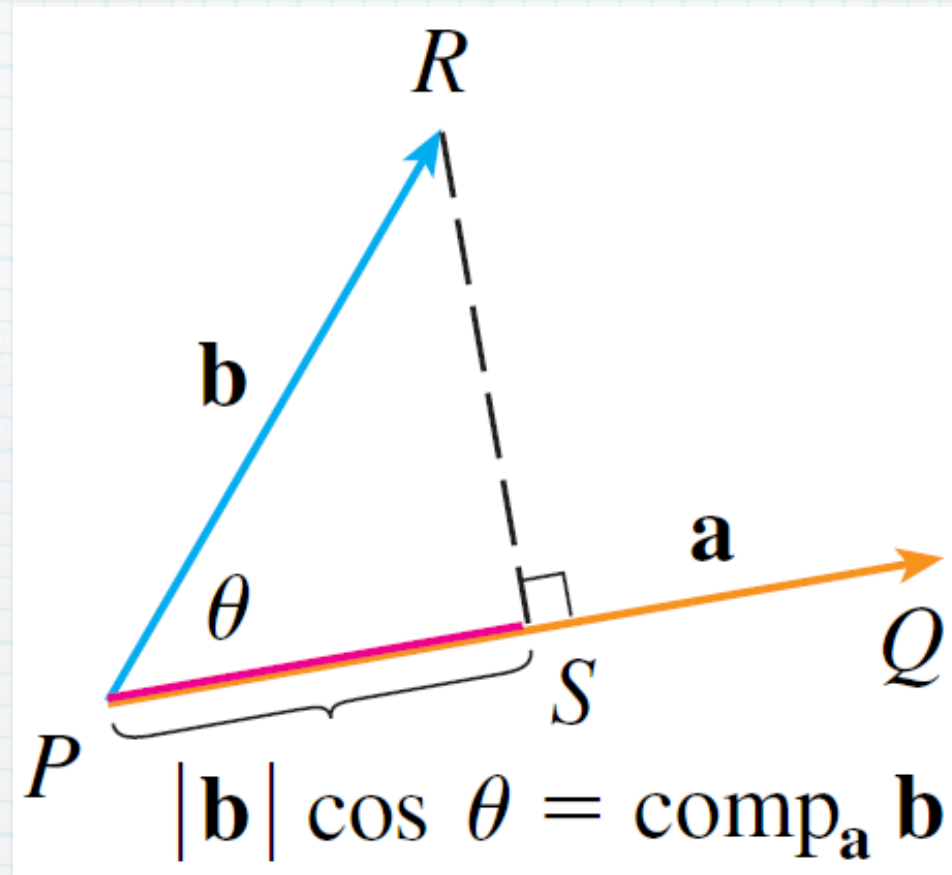
4.  $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

5.  $\mathbf{0} \cdot \mathbf{a} = 0$

# Projections



# Projections



Scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

# Handouts

- \* Regular Hexagons
- \* The Right Stuff



# Cross Product

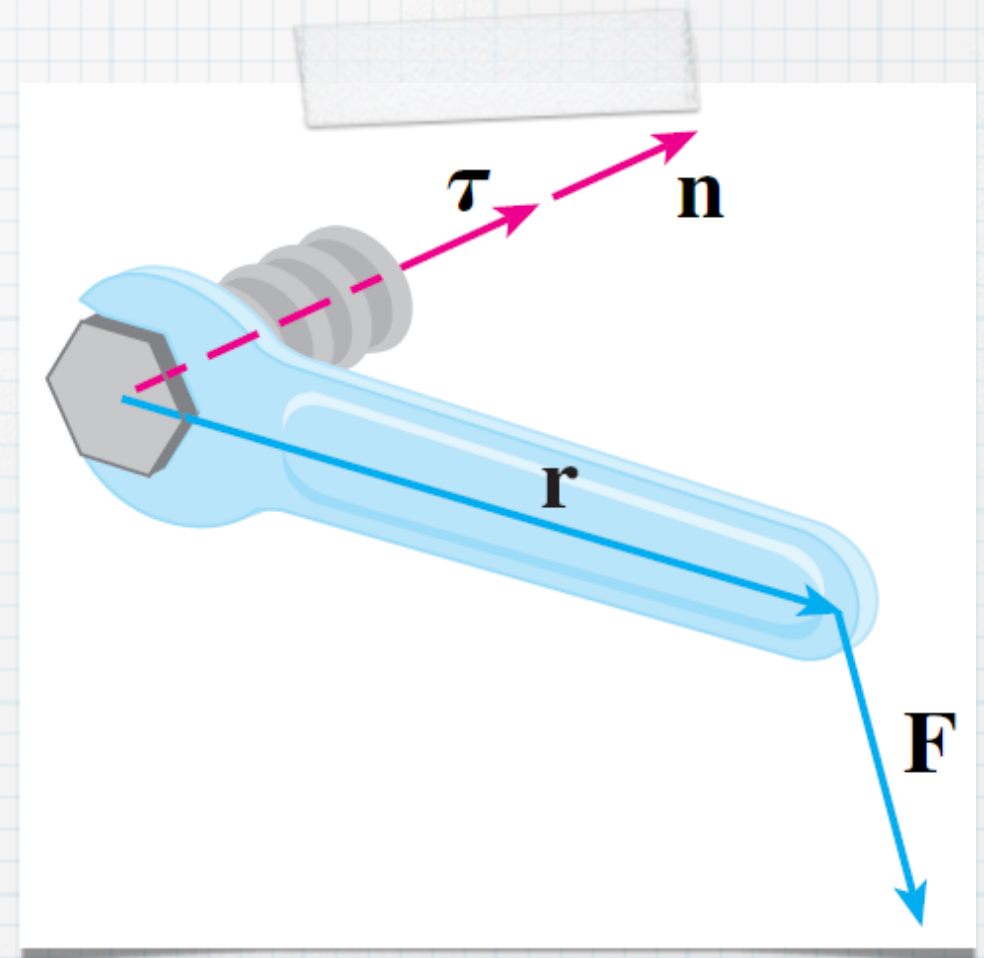
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## Section 9.4

# Torque

Depends on

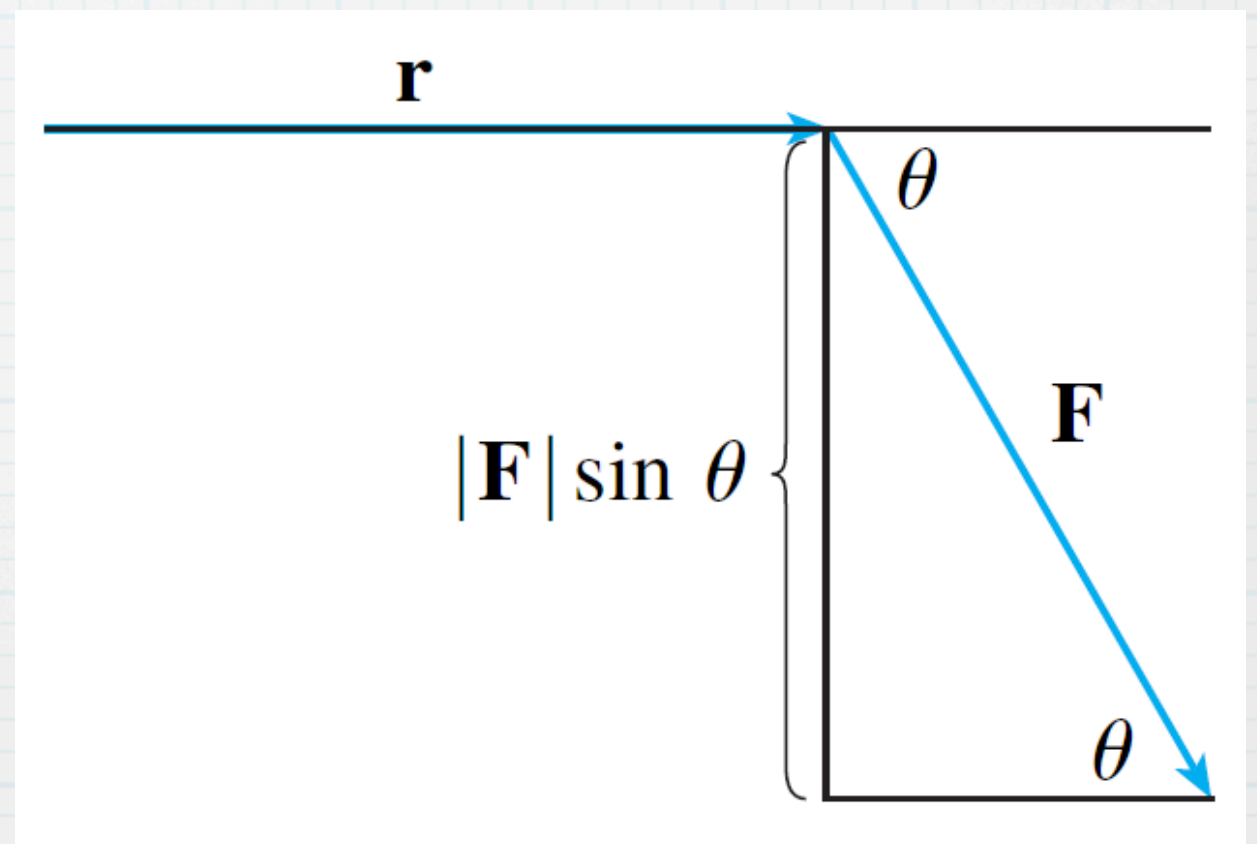
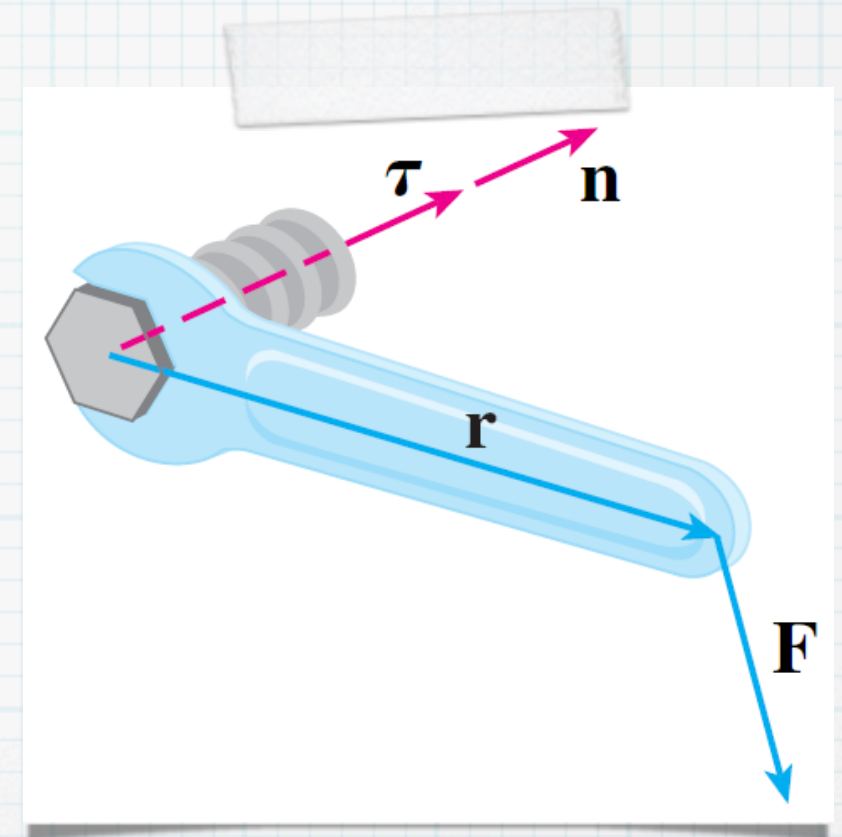
- \* Distance from axis of bolt to point at which Force is applied:  
 $|r|$  = magnitude of vector  $r$
- \* Scalar component of Force in the direction perpendicular to  $r$ .  
This is the only component of the Force which can cause a rotation.



# Torque

Depends on

- \* Distance from axis of bolt to point at which Force is applied:  
 $|r|$  = magnitude of vector  $r$
- \* Scalar component of Force in the direction perpendicular to  $r$ .  
This is the only component of the Force which can cause a rotation.



$$|\tau| = |r| |F| \sin \theta$$

# Cross Product

**Definition** If  $\mathbf{a}$  and  $\mathbf{b}$  are nonzero three-dimensional vectors, the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the vector

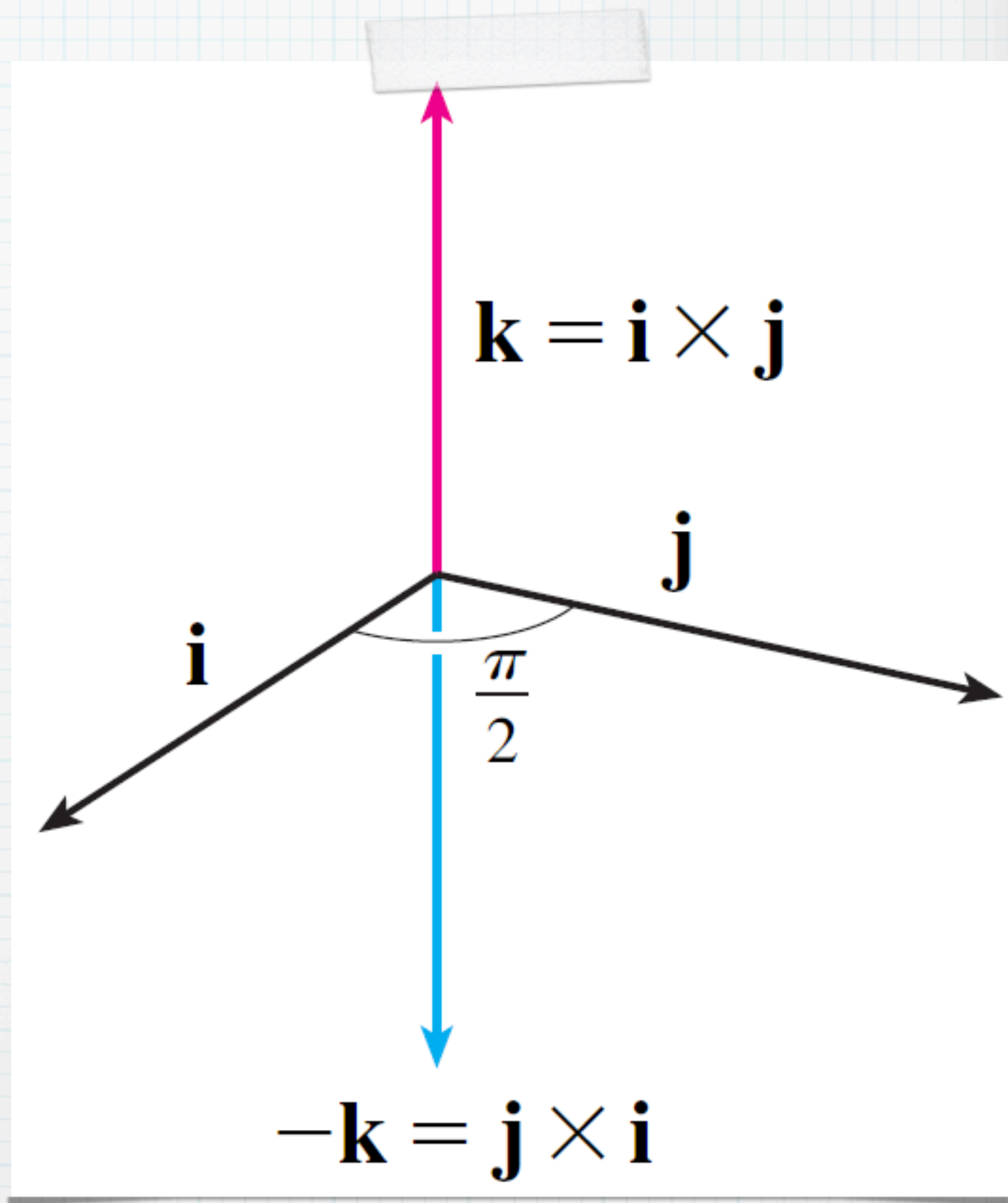
$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}| |\mathbf{b}| \sin \theta) \mathbf{n}$$

where  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$ ,  $0 \leq \theta \leq \pi$ , and  $\mathbf{n}$  is a unit vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  and whose direction is given by the **right-hand rule**: If the fingers of your right hand curl through the angle  $\theta$  from  $\mathbf{a}$  and  $\mathbf{b}$ , then your thumb points in the direction of  $\mathbf{n}$ . (See Figure 3.)



# Cross Product:

## Right Hand Rule

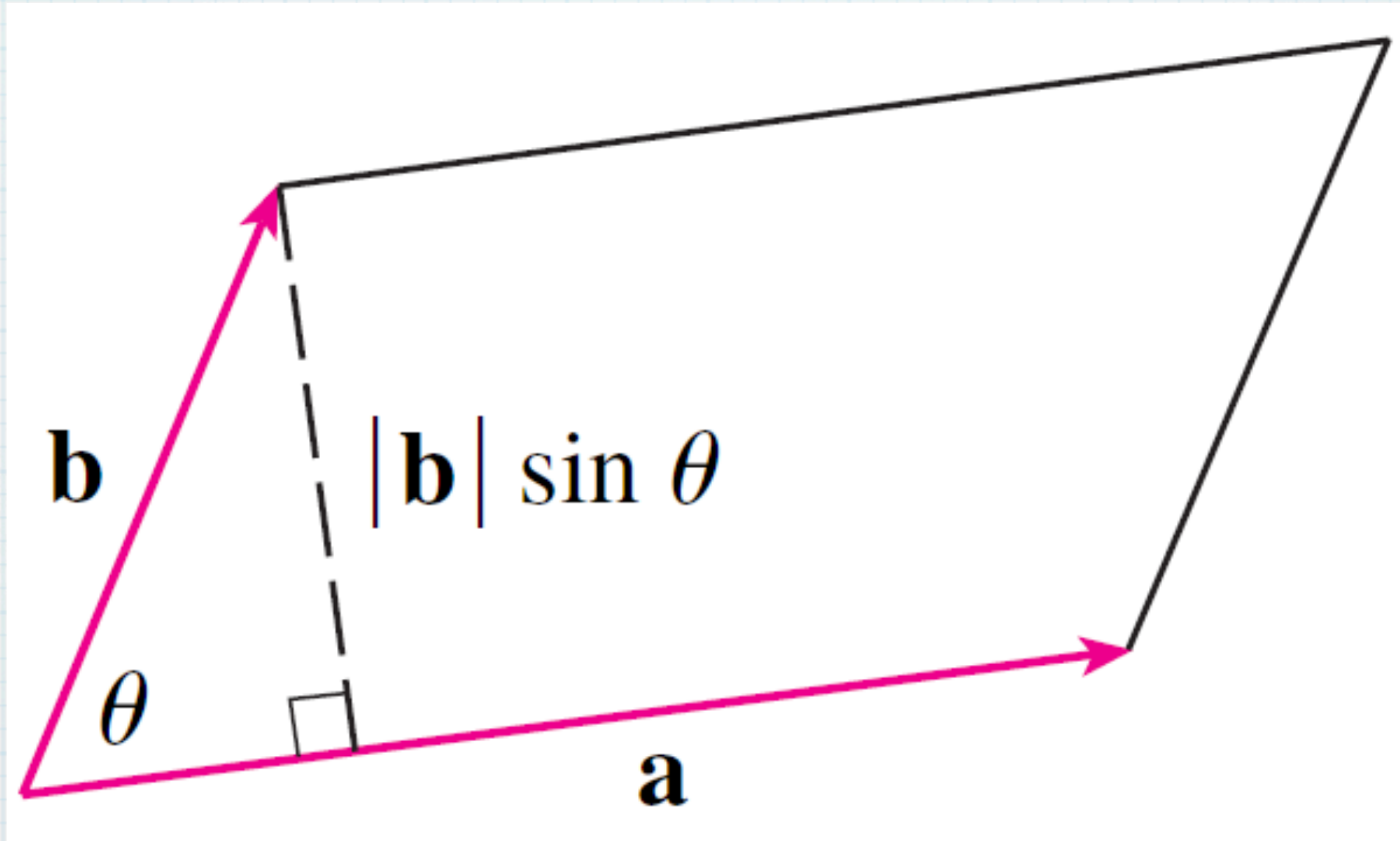


# Cross Product: Properties

**Properties of the Cross Product** If **a**, **b**, and **c** are vectors and  $c$  is a scalar, then

1.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2.  $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3.  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4.  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

# Geometry



The length of the cross product  $\mathbf{a} \times \mathbf{b}$  is equal to the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$ .

# Computing the Cross Product

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$$

$$= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k}$$

$$+ a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k}$$

$$+ a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k}$$

$$= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i})$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

**2** If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

# Computing the Cross Product: Determinant Form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$



# Example

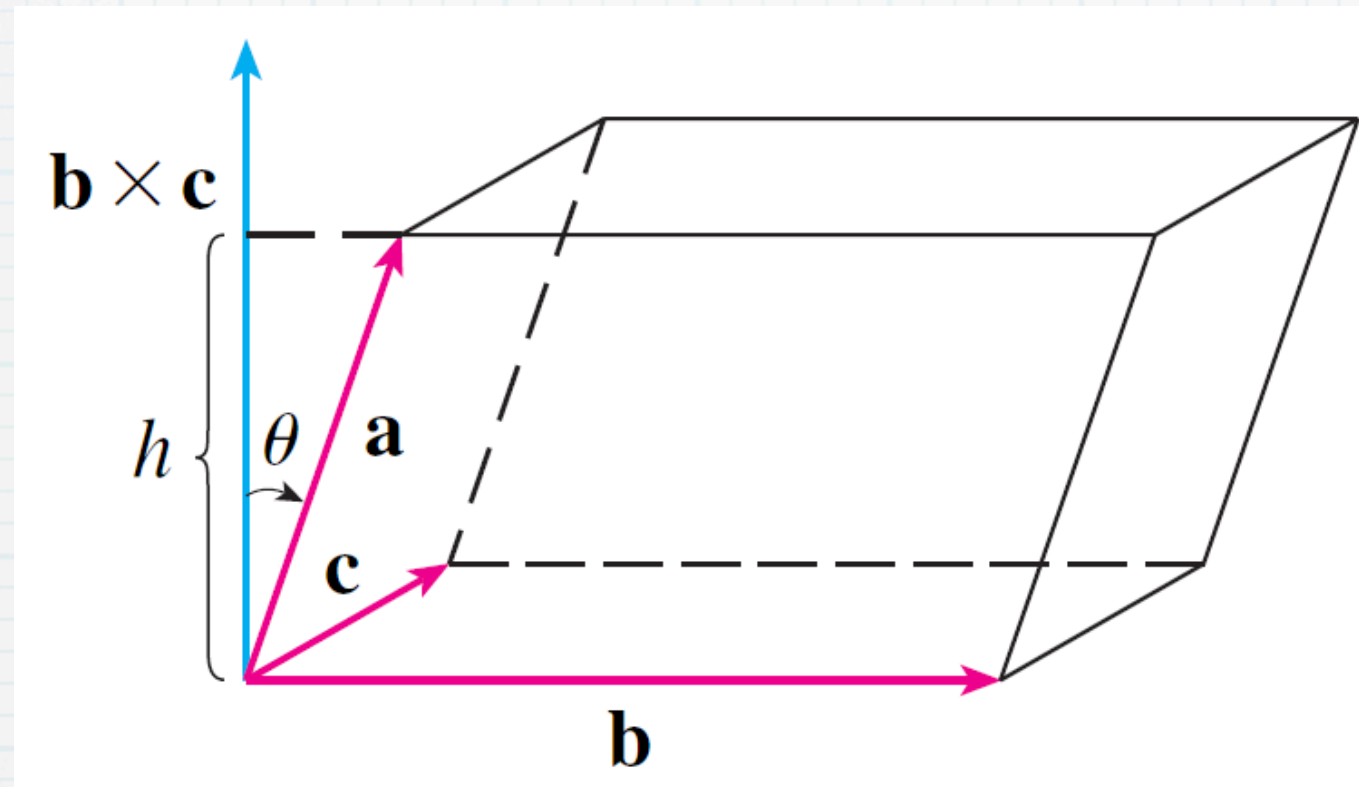
If  $\mathbf{a} = \langle 1, 3, 4 \rangle$  and  $\mathbf{b} = \langle 2, 7, -5 \rangle$  then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix} \mathbf{k}$$

$$= (-15 - 28) \mathbf{i} - (-5 - 8) \mathbf{j} + (7 - 6) \mathbf{k} = -43\mathbf{i} + 13\mathbf{j} + \mathbf{k}$$

# Scalar Triple Product



The volume of the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is the magnitude of their scalar triple product:

$$V = | \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) |$$

# Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot \left[ \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \mathbf{k} \right]$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

# Vector Triple Product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

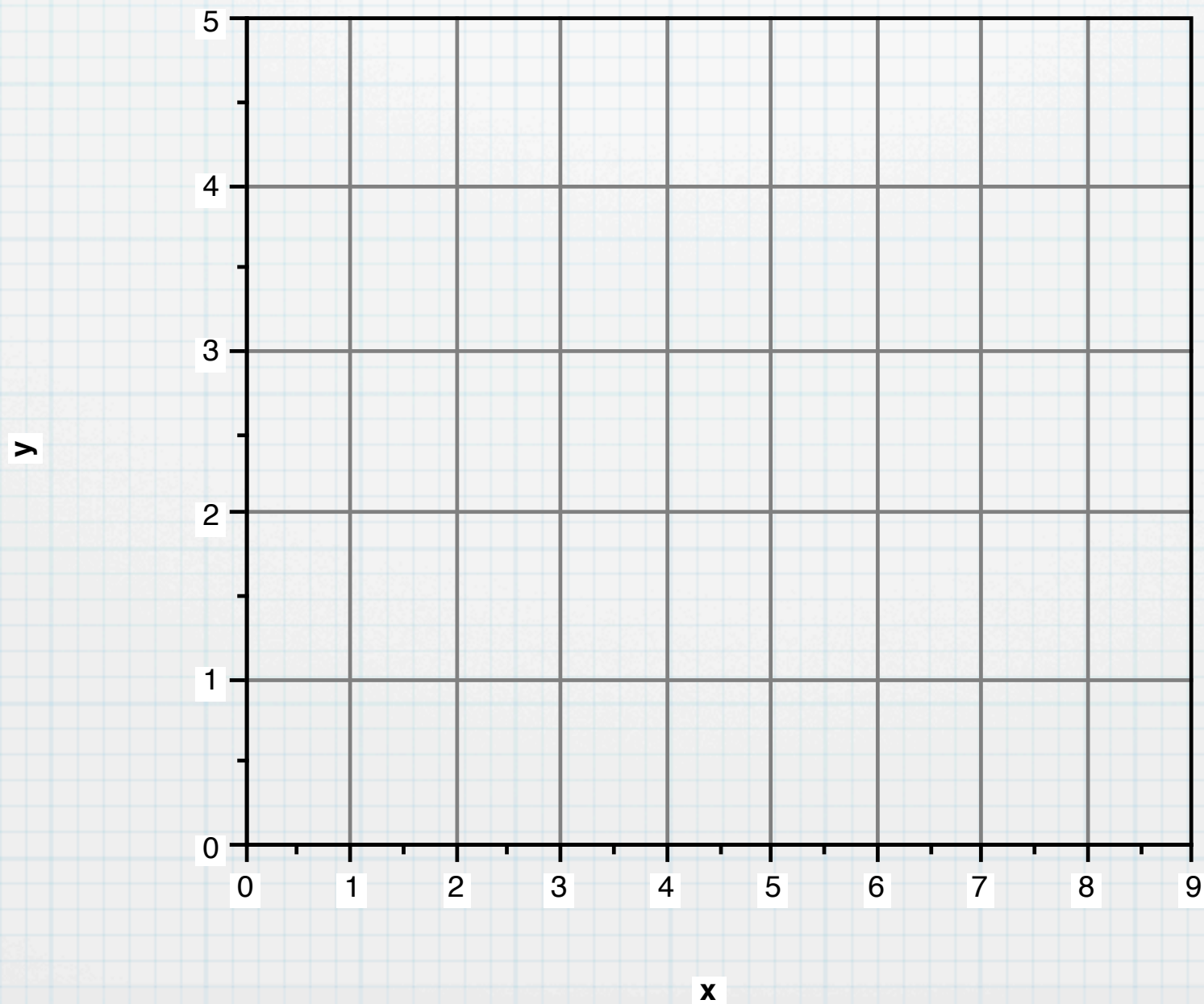
# Handouts

- \* Dot and Cross Product
- \* Messing with the Cross Product



# Preview: Lines

$$\mathbf{r}(t) = \langle 4, 3 \rangle + t \langle 2, -1 \rangle$$

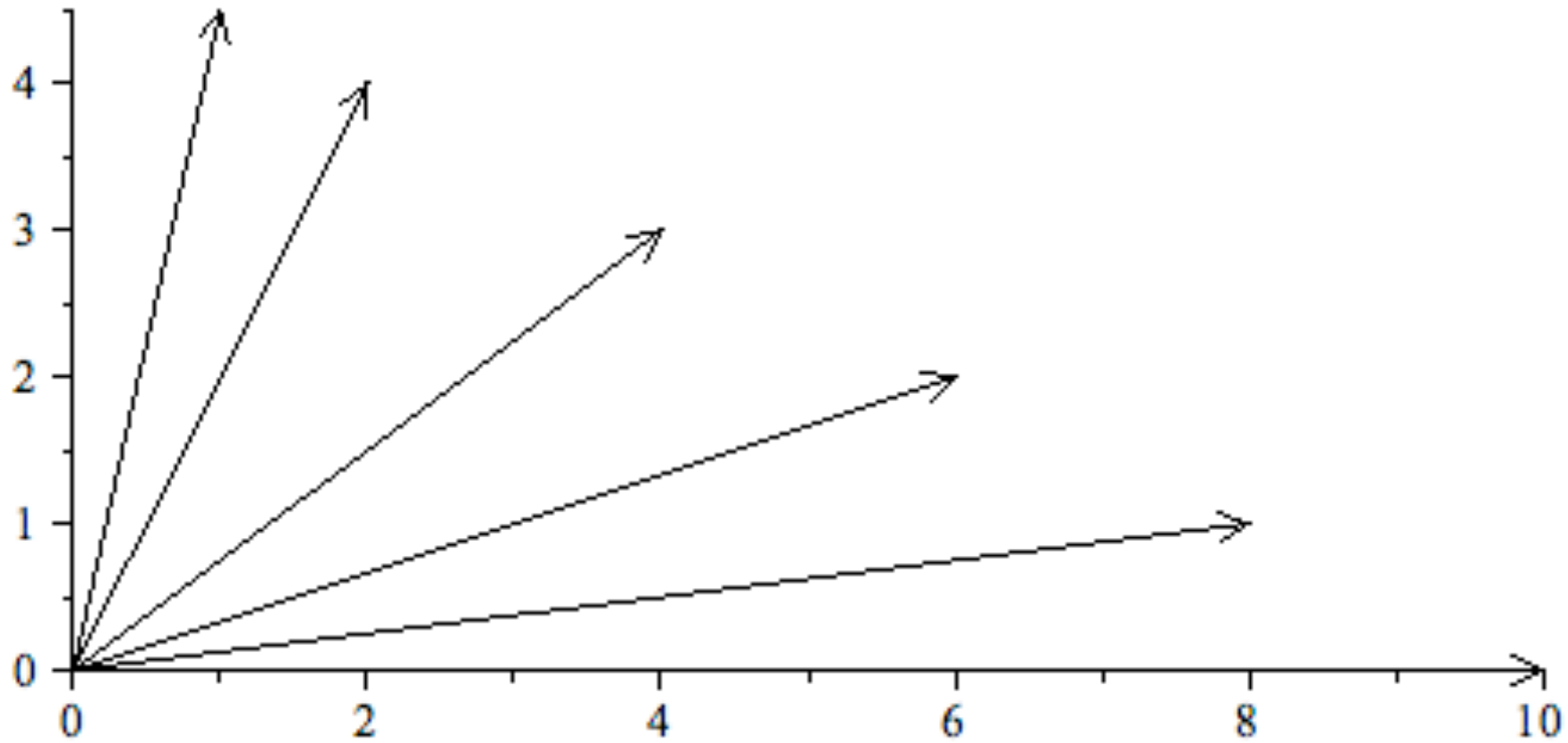


# Handout

\* Lines in the Plane

# Lines

$$\mathbf{r}(t) = \langle 4, 3 \rangle + t \langle 2, -1 \rangle$$



# Homework

- \* WebAssign Assignment 2
- \* Mathematica Assignment 2